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LETTER TO THE EDITOR

# The exchange algebra for Liouville field on Riemann surfaces with $h > 1$ genus†

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**Abstract.** We consider the classical Liouville field theory on the Riemann surface with $h > 1$ genus. In terms of the uniformization theorem of the Riemann surface, we show explicitly the classical exchange algebra (CEA) for the chiral components of the Liouville field. We find that this classical exchange algebra is just the same as that on a Riemann sphere with $2h$ punctures.

The Liouville theory has attracted much attention over a long time. Early interest in it rests mainly on the uniformization theory of the two-dimensional Riemann sphere [1, 2]. Recently, much attention has again been paid to the classical and quantum Liouville theory [3], since this theory plays an important role in string theory (ST), 2D quantum gravity (QG) and conformal field theory (CFT), and is related to quantum groups. The structure of the Poisson bracket algebras or classical exchange algebra (CEA) of Liouville theory on a cylinder has already been explored by Gervais and Neveu and others [4, 5]. In order to match the study of ST, QG and CFT, it is necessary to find out the properties of the exchange algebraic structure for the classical Liouville theory on Riemann surfaces. In [6], we have mainly studied the classical and quantum Liouville theory on the Riemann sphere with $n > 3$ punctures and we have obtained some interesting results. In this letter, we will generalize the classical exchange algebra in [6] to the case of the Riemann surface with $h > 1$ genus. We will review the problem of uniformization of the Riemann surface, and study monodromy group and exchange algebra. Finally, we will give some conclusions and remarks.

A marked Riemann surface of genus $h > 1$ is a compact Riemann surface $X$ with $x_0 \in X$ taken as fixed, together with a choice of a set of generators $\alpha_i$, $\beta_i$ $(i = 1, \ldots, h)$ of fundamental group $\pi_1(X, x_0)$, and these generators satisfy the relation

$$\Pi_{i=1}^{h} \alpha_i^{-1} \beta_i^{-1} \alpha_i \beta_i = 1. \tag{1}$$

A dissection of $X$ can be performed by cutting off $X$ along $2h$ homology cycles starting from the point $x_0$. The result is a planar polygon in a subregion of the (extended) complex

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plane. The reverse procedure of dissection, which identifies edges of the polygon by the prescribed generators $\alpha_i$, $\beta_i$, is what the famous uniformization theorem amounts to. The mathematically rigorous way of uniformizing a surface is to take a covering space $\Omega$ and a set of covering maps. Generally speaking, there are two ways of uniformization, which correspond to the Fuchsian uniformization and the Schottky uniformization. In this letter, we choose the Fuchsian uniformization.

The Fuchsian uniformization is defined as a set of covering maps $\pi_\Gamma : H \to X$, or in other words $X = H/\Gamma$, where $H = \{z \in C \mid I_m z > 0\}$ is the upper half plane and its automorphism group $\Gamma \subset PSL(2, R)$ is a strictly hyperbolic Fuchsian group ($\Gamma$ acts on $H$ by linear fractional transformation) and $\Gamma$ is isomorphic to the fundamental group $\pi_1(X, x_0)$ of $X$. Denoting by $A_i$, $B_i$ $(i = 1, \ldots, h)$ the generators of $\Gamma$ corresponding to the elements $\alpha_i$, $\beta_i$ $(i = 1, \ldots, h) \in \pi_1(X, x_0)$, obviously

$$\Pi_{i=1}^{h} A_i^{-1} B_i^{-1} A_i B_i = 1. \tag{2}$$

The group $\Gamma$ with the distinguished system of generators $A_i$, $B_i$ is called the marked Fuchsian group corresponding to the marked Riemann surface $(X; \alpha_i, \beta_i)$.

We start with Liouville's action on a compact Riemann surface $X$ of $h > 1$ genus [3],

$$S[\psi, \hat{g}] = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[ \tfrac{1}{2}\psi \Delta_{\hat{g}} \psi + k_0 R_{\hat{g}} \psi + \frac{\mu^2}{2\alpha_0^2} \mathrm{e}^{2\alpha_0 \psi} \right] \tag{3}$$

whose equation of motion is given by

$$\Delta_{\hat{g}} \psi + k_0 R_{\hat{g}} + \frac{\mu^2}{\alpha_0} \mathrm{e}^{2\alpha_0 \psi} = 0. \tag{4}$$

Here $\hat{g}$, which depends only on $3h - 3$ complex moduli parameters of Riemann surfaces, is related to a general metric $g$ on Riemann surface as

$$g = \mathrm{e}^{2\sigma} \hat{g}.$$

Their curvatures are then related by

$$\sqrt{g} R_g = \sqrt{\hat{g}} R_{\hat{g}} + \sqrt{\hat{g}} \Delta_{\hat{g}} \sigma. \tag{5}$$

On any compact Riemann surface $X$, there exists a constant negative curvature $-\mu^2$, so equation (5) can reduce to an equation for a function $\sigma(\xi)$ that Weyl rescales a given metric $\hat{g}$ to constant curvature $g$

$$\Delta_{\hat{g}} \sigma + R_{\hat{g}} + \mu^2 \mathrm{e}^{2\sigma} = 0. \tag{6}$$

If we take $\sigma = \alpha_0 \psi$ and let $\alpha_0 k_0 = 1$, then equation (6) coincides with the Liouville equation (4).

We may always take a conformal flat metric locally on the Riemann surface, such that

$$\partial_\omega \partial_{\bar{\omega}} \sigma + \mathrm{e}^{2\sigma} = 0 \tag{7}$$

where we have taken $\mu^2 = 1$. If we take $\alpha_0 = \frac{1}{2}$, by taking a conformal flat metric, we can get the following equation from (4)

$$\partial_\omega \partial_{\bar{\omega}} \psi + 2\mathrm{e}^{\psi} = 0. \tag{8}$$

The corresponding momentum–energy tensor of the Liouville field is

$$T(\psi) = \partial_\omega \partial_\omega \psi - \tfrac{1}{2}(\partial_\omega \psi)^2. \tag{9}$$

If we choose $e^\psi$ as a (1, 1)-form on the Riemann surface, then the form of this equation is independent of the choice of local coordinate [7]. For the complex atlas on $X:(U_\alpha, \omega_\alpha) \to (U_\beta, \omega_\beta)$ with holomorphic coordinate change $\omega_\alpha = f_{\alpha\beta}(\omega_\beta)$, equation (8) will not change its form if

$$\psi_\beta(\omega_\beta) = \psi_\alpha(f_{\alpha\beta}(\omega_\beta)) + \log |f'_{\alpha\beta}(\omega_\beta)|^2. \tag{10}$$

We can consider it as the Kähler form of a metric on $X$ if $e^\psi$ is regular. It is possible to find the solution of (8) such that $e^\psi$ is a (1,1)-form from the viewpoint of uniformization.

According to the Fuchsian uniformization, we can define a conformal map between the half upper plane and Riemann surface $X$

$$\pi_\Gamma : H \to X$$

so that $\pi_\Gamma(z) = \omega$ and $\pi_\Gamma(\Gamma z) = \pi_\Gamma(z)$; $\omega \in X$, $z \in H$; here $\Gamma$ is a strictly hyperbolic Fuchsian group. The $\pi_\Gamma^{-1}(\omega)$ is a collection of the local univalent linearly polymorphic function (which means this function is a locally Schlicht and locally meromorphic function which transforms linear fractionally under the fundamental group) on $X$.

Using the $\pi_\Gamma^{-1}(\omega)$, we can construct properly the solution of (7) or (8):

$$e^{2\sigma} = e^\psi = \frac{|(\pi_\Gamma^{-1})'(\omega)|^2}{(I_m \pi_\Gamma^{-1}(\omega))^2}. \tag{11}$$

We find that

$$\mathcal{D}(\pi_\Gamma^{-1}) = T(\psi) \tag{12}$$

where $\mathcal{D}(f) = (f'''/f') - \frac{3}{2}(f''/f')^2$ is the Schwarzian derivative. After defining the projective structure on $X$, the function $\pi_{\Gamma,\alpha}^{-1}$ on Chart $U_\alpha \subset X$ is related to $\pi_{\Gamma,\beta}^{-1}$ on Chart $U_\beta \subset X$ by

$$\pi_{\Gamma,\alpha}^{-1}(\omega_\alpha) = \frac{a_{\alpha\beta}\pi_{\Gamma,\beta}^{-1}(\omega_\beta) + b_{\alpha\beta}}{c_{\alpha\beta}\pi_{\Gamma,\beta}^{-1}(\omega_\beta) + d_{\alpha\beta}} \qquad a_{\alpha\beta}b_{\alpha\beta} - c_{\alpha\beta}d_{\alpha\beta} = 1.$$

This means that the collection $\{\pi_{\Gamma,\alpha}^{-1}\}$ is a section of a flat $PSL(2, R)$ bundle on $X$. It is easy to prove that the solution (11) is a (1, 1)-form and it is invariant when $\pi_{\Gamma,\alpha}^{-1}(\omega)$ is changed under the linear fractional transformation.

It is well known that the uniformization problem of the Riemann surface is related to the Fuchsian equation:

$$\frac{d^2\eta(\omega)}{d\omega^2} + \tfrac{1}{2}Q_x(\omega)\eta(\omega) = 0 \tag{13}$$

where $Q_x(\omega)$ is related to $3h - 3$ accessory parameters of the Fuchsian uniformization of $X$ and it is transformed as the Schwarzian derivation as the local chart is changed. $\eta$ is understood as a (multi-valued) differential on $X$ of order $-\frac{1}{2}$. It makes sense to speak of solutions of the Fuchsian equation defined globally on a Riemann surface $X$,

$$\frac{d^2\eta(P)}{dP^2} + \tfrac{1}{2}Q_x(P)\eta(P) = 0 \tag{14}$$

for all points $P \in X$. These are topologically inequivalent solutions indexed by $2h$ homology bases of $X$. It was proved in [8] that such solutions depend on the parameters of monodromy group.

*$Q_x(\omega)$ can be realized as $\mathcal{D}(\pi_\Gamma^{-1})$. In this case, $\pi_\Gamma^{-1} = \eta_2/\eta_1$, where $\eta_1$ and $\eta_2$ are two* linearly independent solutions of (13). As a conclusion, the solution $\psi$ of (8) is constructed in such a way that it also depends on the $3h-3$ accessory parameters. The relations between accessory parameters and moduli parameters are very complicated, but they have been discussed in the mathematical literature [2]. In this sense we may think that the solution (11) in fact includes information about the global property of the Riemann surface.

Using the characteristics of the meromorphic differential on the Riemann surface [9], we introduce the local coordinate $(\sigma, \tau)$. The Liouville equation can be written as

$$\psi_{\sigma\sigma} - \psi_{\tau\tau} - 2e^{\psi} = 0. \tag{15}$$

From equation (12), we get

$$\mathcal{D}(\pi_\Gamma^{-1}) = \psi_{\omega\omega} - \tfrac{1}{2}\psi_\omega^2 = \tfrac{1}{4}\psi_{\sigma\sigma} + \tfrac{1}{2}\psi_{\sigma\tau} + \tfrac{1}{4}\psi_{\tau\tau} - \tfrac{1}{8}(\psi_\sigma + \psi_\tau)^2. \tag{16}$$

Starting from the Liouville action, we can get the canonical conjugate momentum of the Liouville field $\pi = \delta S/\delta\psi = \psi_\tau$ and it satisfies the Poisson bracket

$$\begin{aligned} &\{\psi(\sigma,\tau), \pi(\sigma',\tau)\} = \delta(\sigma - \sigma') \\ &\{\psi(\sigma,\tau), \psi(\sigma',\tau)\} = \{\pi(\sigma,\tau), \pi(\sigma',\tau)\} = 0. \end{aligned} \tag{17}$$

We get

$$\{U(\sigma), U(\sigma')\} = -\tfrac{1}{8}\delta'''(\sigma - \sigma') + \tfrac{1}{4}(\partial_\sigma - \partial_{\sigma'})U(\sigma)\delta(\sigma - \sigma') \tag{18}$$

where

$$U = \mathcal{H} + \mathcal{P} = -\tfrac{1}{2}\mathcal{D}(\pi_{\Gamma-1}) \qquad \mathcal{H} = \tfrac{1}{16}\pi^2 + \tfrac{1}{16}\psi_\sigma^2 + \tfrac{1}{8}e^{\psi} - \tfrac{1}{8}\psi_{\sigma\sigma} \qquad \mathcal{P} = \tfrac{1}{8}\pi\psi_\sigma - \tfrac{1}{4}\pi_\sigma.$$

The $U$ plays the role of Liouville energy–momentum tensor; in other words, this system is integrable, and (18) can be regarded as the realization of Virasoro algebra in the classical case. Let

$$A = \pi_\Gamma^{-1} \qquad \delta K = -\tfrac{1}{2}\partial_\sigma \log A_\sigma \qquad P = -\tfrac{1}{2}\log A_\sigma \qquad K^2 + K_\sigma = U(\sigma).$$

After some calculation, we find

$$\begin{aligned} &\{K(\sigma), K(\sigma')\} = \tfrac{1}{16}(\partial_\sigma - \partial_{\sigma'})\delta(\sigma - \sigma') \\ &\{P(\sigma), P(\sigma')\} = -\tfrac{1}{16}\varepsilon(\sigma - \sigma') \\ &\{A(\sigma), A(\sigma')\} = \tfrac{1}{8}\varepsilon(\sigma - \sigma')(A(\sigma) - A(\sigma'))^2 + \tfrac{1}{8}(A^2(\sigma) - A^2(\sigma')) \end{aligned} \tag{19}$$

where $\varepsilon(\sigma - \sigma')$ is a signal function. To solve the Fuchsian equation (or Schrödinger equation with zero eigenvalue)

$$\frac{\mathrm{d}^2\eta(\sigma)}{\mathrm{d}\sigma^2} + \tfrac{1}{2}\mathcal{D}(A)\eta(\sigma) = 0 \tag{20}$$

we get $\eta_1 = 1/\sqrt{A_\sigma}$, $\eta_2 = A/\sqrt{A_\sigma}$, $A = \eta^2/\eta^1$. By (19), we can obtain the Poisson bracket of $n_i$ $(i = 1, 2)$

$$\{\eta_i(\sigma), \eta_j(\sigma')\} = S_{ij}^{kl}\eta_k(\sigma)\eta_l(\sigma') \tag{21}$$

where $i, j, k, l = 1, 2$ and $S_{ij}^{kl} = -\frac{1}{4}[r_+\theta(\sigma - \sigma') + r_-\theta(\sigma' - \sigma)]|_{ij}^{kl}$ and $\theta(\sigma)$ is the step function, the $4 \times 4$ matrices $r_\pm$ are the solutions of the classical Y–B equation and can be expressed by the generators of the Lie algebra $sl(2, r)$:

$$\begin{aligned} &r_\pm = \pm H \otimes H \pm 4E_\pm \otimes E_\mp \\ &[H, E_\pm] = \pm 2E_\pm \qquad [E_+, E_-] = H. \end{aligned} \tag{22}$$

We know that the projective monodromy group $M$ related to Fuchsian equation (20) belongs to $PSL(2, R)$. Let $\{S_i\}$, $i = 1, \ldots, 2h$ be the set of linear fractional transformations, $S_i \in M$, then

$$S_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \qquad \alpha_i\delta_i - \beta_i\gamma_i = 1 \qquad \alpha_i + \delta_i \neq 2 \qquad (i = 1, \ldots, 2h).$$

We find that if the elements of matrix $S_i$ is real, Liouville field $\psi(P)$ is periodic for a closed path $C_\lambda$ around any genus (there are two different path per genus) on $X$ when the analytic continuation of a pair of solution $\eta_1$ and $\eta_2$ around $C_\lambda$ results in a pair of new solutions $\eta_1^\lambda$ and $\eta_2^\lambda$, and

$$\begin{pmatrix} \eta_2^\lambda \\ \eta_1^\lambda \end{pmatrix} = (S_\lambda)\begin{pmatrix} \eta_2 \\ \eta_1 \end{pmatrix}.$$

That the Liouville field $\psi(\sigma)$ and its conjugate $\pi(\sigma)$ are smooth and single-valued around any genus on $X$ enables us to make the following assumptions:

$$\{\psi(P), \pi(P')\} = \{\psi(P + C_\lambda), \pi(P' + C_\lambda)\} = \Delta(P - P') \tag{23}$$

and if we assume this transformation becomes a group, then

$$\{\psi(P), \pi(P')\} = \{\psi(P + C_\lambda + C_\rho), \pi(P' + C_\lambda + C_\rho)\} = \Delta(P - P') \tag{24}$$

where $P$, $P' \in C_\tau$ and $C_\tau$ is the level curve, and $\Delta(P - P')$ is the delta function on $C_\tau$. In terms of (23) and (24), we can generalize (21) to

$$\{\eta_i^\lambda(P), \otimes\eta_j^\lambda(P')\} = S_{ij}^{kl}\eta_k^\lambda(P)\eta_l^\lambda(P') \tag{25}$$

$$\{\eta_i^{\lambda\rho}(P), \otimes\eta_j^{\lambda\rho}(P')\} = S_{ij}^{kl}\eta_k^{\lambda\rho}(P)\eta_l^{\lambda\rho}(P') \tag{26}$$

where $i, j, k, l = 1, 2$; $\lambda, \rho = 1, \ldots, 2h$ and

$$\begin{pmatrix} \eta_2^{\lambda\rho} \\ \eta_1^{\lambda\rho} \end{pmatrix} = (S_\rho)(S_\lambda) \begin{pmatrix} \eta_2 \\ \eta_1 \end{pmatrix}. \tag{27}$$

From (25) and (26), we can get the following Poisson brackets

$$\begin{aligned}
&\{\alpha_\lambda, \eta_1\} = \frac{\gamma_\lambda}{8}\eta_2 \qquad \{\alpha_\lambda, \eta_2\} = \frac{\beta_\lambda}{8}\eta_1 \\
&\{\beta_\lambda, \eta_1\} = -\frac{\alpha_\lambda - \delta_\lambda}{8}\eta_2 \qquad \{\beta_\lambda, \eta_2\} = 0 \\
&\{\gamma_\lambda, \eta_1\} = 0 \qquad \{\gamma_\lambda, \eta_2\} = -\frac{\alpha_\lambda - \delta_\lambda}{8}\eta_1 \\
&\{\delta_\lambda, \eta_1\} = -\frac{\gamma_\lambda}{8}\eta_2 \qquad \{\alpha_\lambda, \eta_2\} = -\frac{\beta_\lambda}{8}\eta_1
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\{\alpha_\lambda, \alpha_\rho\} = \tfrac{1}{8}(\beta_\lambda\gamma_\rho - \beta_\rho\gamma_\lambda) \qquad \{\alpha_\lambda, \beta_\rho\} = -\tfrac{1}{8}\beta_\lambda(\alpha_\rho - \delta_\rho) \\
&\{\alpha_\lambda, \gamma_\rho\} = \tfrac{1}{8}\gamma_\lambda(\alpha_\rho - \delta_\rho) \qquad \{\alpha_\lambda, \delta_\rho\} = -\tfrac{1}{8}(\beta_\lambda\gamma_\rho - \beta_\rho\gamma_\lambda) \\
&\{\beta_\lambda, \beta_\rho\} = 0 \qquad \{\beta_\lambda, \gamma_\rho\} = -\tfrac{1}{8}(\alpha_\lambda - \delta_\lambda)(\alpha_\rho - \delta_\rho) \\
&\{\beta_\lambda, \delta_\rho\} = -\tfrac{1}{8}(\alpha_\lambda - \delta_\lambda)\beta_\rho \qquad \{\gamma_\lambda, \gamma_\rho\} = 0 \\
&\{\gamma_\lambda, \delta_\rho\} = \tfrac{1}{8}(\alpha_\lambda - \delta_\lambda)\gamma_\rho \qquad \{\delta_\lambda, \delta_\rho\} = \tfrac{1}{8}(\beta_\lambda\gamma_\rho - \beta_\rho\gamma_\lambda).
\end{aligned} \tag{29}$$

The elements of the monodromy matrices are dynamical variables in our case. When they satisfy the condition

$$\frac{\alpha_\lambda}{\gamma_\lambda} = \frac{\alpha_\rho}{\gamma_\rho} \qquad \frac{\beta_\lambda}{\gamma_\lambda} = \frac{\beta_\rho}{\gamma_\rho} \qquad \frac{\delta_\lambda}{\gamma_\lambda} = \frac{\delta_\rho}{\gamma_\rho} \tag{30}$$

the Jacobi identity will be satisfied at the same time. By the non-trivial properties shown in (28) of the elements of monodromy matrices, we may consider that all of the $2h$ solutions $\eta_i^\lambda(P)$, $i = 1, 2$; $\lambda = 1, \ldots, 2h$, are independent.

After some calculation, the same Poisson bracket for $\eta_i^\lambda(P)$, as in the punctured sphere case, is found to be

$$\{\eta_i^\lambda(P), \eta_i^\rho(P')\} = -\tfrac{1}{16}\varepsilon(P - P')[2\eta_i^\rho(P)\eta_i^\lambda(P') - \eta_i^\lambda(P)\eta_i^\rho(P')] \tag{31}$$

where $i = 1, 2$, and

$$\begin{aligned}
&\{\eta_1^\lambda(P), \eta_2^\rho(P')\} = \tfrac{1}{16}\varepsilon(P - P')[\eta_1^\lambda(P)\eta_2^\rho(P') - 2\eta_2^\rho(P)\eta_1^\lambda(P')] - \tfrac{1}{8}\eta_2^\lambda(P)\eta_1^\rho(P') \\
&\{\eta_2^\lambda(P), \eta_1^\rho(P')\} = \tfrac{1}{16}\varepsilon(P - P')[\eta_2^\lambda(P)\eta_1^\rho(P') - 2\eta_1^\rho(P)\eta_2^\lambda(P')] + \tfrac{1}{8}\eta_1^\lambda(P)\eta_2^\rho(P').
\end{aligned} \tag{32}$$

One can check that equations (31), (32) coincide with (21) if $\lambda = \rho$. We arrange $\eta_i^\lambda$, $i = 1, 2$, $\lambda = 1, \ldots, 2h - 2$, into a vector

$$\Psi(P) = (\eta_1^1(P), \eta_2^1(P), \eta_1^2(P), \eta_2^2(P), \ldots, \eta_1^{2h-2}(P), \eta_2^{2h-2}(P))$$

and the Poisson brackets can be written in the standard form

$$\{\Psi_l(P), \otimes \Psi_m(P')\} = -\tfrac{1}{4} \sum_{l'k'} [(R_+)_{lm}^{l'm'} \theta(P - P') + (R_-)_{lm}^{l'm'} \theta(P' - P)] \Psi_{l'}(P) \otimes \Psi_{m'}(P') \tag{33}$$

(i) when $l, m$ are both odd or even

$$(R_+)_{lm}^{l'm'} = -\tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} + \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} \qquad (R_-)_{lm}^{l'm'} = \tfrac{1}{4}\delta_l^{l'} - \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} \tag{34}$$

(ii) when $l$ is odd and $m$ is even

$$(R_+)_{lm}^{l'm'} = -\tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} + \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} + \tfrac{1}{2}\delta_{l'}^{l+1}\delta_{m-1}^{m'} \qquad (R_-)_{lm}^{l'm'} = \tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} - \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} + \tfrac{1}{2}\delta_{l'}^{l+1}\delta_{m-1}^{m'} \tag{35}$$

(iii) when $l$ is even and $m$ is odd

$$(R_+)_{lm}^{l'm'} = -\tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} - \tfrac{1}{2}\delta_{l'}^{l-1}\delta_{m+1}^{m'} \qquad (R_-)_{lm}^{l'm'} = \tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} - \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} - \tfrac{1}{2}\delta_{l'}^{l-1}\delta_{m+1}^{m'} \tag{36}$$

and $R_+ - R_- = C_{2k\otimes 2k} - \frac{k-1}{2k} I_{2k\otimes 2k}$, where $C_{2k\otimes 2k}$ is the Casimir operator of $SL(2k, R) \otimes SL(2k, R)$, and $I_{2k\otimes 2k}$ is the unity matrix ($k = 2h - 2$). It shows that this exchange matrix is independent of the parameters of the monodromy group. This matrix is related to the symmetry of the Lie group $SL(n)$ in a non-trivial manner, and is the same as the matrix [6] which dominates the CEA of the Liouville field on the Riemann sphere with $n$ punctures when $n = 2h$. This shows that the exchange matrix of the Liouville field on the Riemann surface depends only on the numbers of non-homotopic path.

In this letter, we have discussed the relation between the solutions of the Fuchsian equation and the solution of the Liouville equation, as well as the monodromy group on the Riemann surface with $h > 1$ genus. We show explicitly the exchange algebra for the chiral components of the Liouville field, and find that this exchange algebra is just the same as the case of a Riemann sphere with $2h$ punctures [6]. This result is very interesting, because it means the exchange algebra is only related to the topological invariant quantity of Riemann surface. Are there other forms of Poisson brackets of monodromy group parameters? It is already known that there exists the so-called dressing symmetry in many nonlinear integrable systems, such as Liouville field theory [10, 11]. It is interesting to ask what is the relation between dressing symmetry and our monodromy transformation in the general case of $S_\lambda \in PSL(2, C)$? Finally, the quantization of Liouville field theory on high-genus Riemann surfaces should be studied further. We will discuss these problems elsewhere.

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